

Superposition of Elliptic Functions as Solutions For a Large Number of Nonlinear Equations

Avinash Khare

Raja Ramanna Fellow, Indian Institute of Science Education and Research (IISER), Pune, India 411021

Avadh Saxena

Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos,
NM 87545, USA

Abstract:

For a large number of nonlinear equations, both discrete and continuum, we demonstrate a kind of linear superposition. We show that whenever a nonlinear equation admits solutions in terms of both Jacobi elliptic functions $\text{cn}(x, m)$ and $\text{dn}(x, m)$ with modulus m , then it also admits solutions in terms of their sum as well as difference. We have checked this in the case of several nonlinear equations such as the nonlinear Schrödinger equation, MKdV, a mixed KdV-MKdV system, a mixed quadratic-cubic nonlinear Schrödinger equation, the Ablowitz-Ladik equation, the saturable nonlinear Schrödinger equation, $\lambda\phi^4$, the discrete MKdV as well as for several coupled field equations. Further, for a large number of nonlinear equations, we show that whenever a nonlinear equation admits a periodic solution in terms of $\text{dn}^2(x, m)$, it also admits solutions in terms of $\text{dn}^2(x, m) \pm \sqrt{m}\text{cn}(x, m)\text{dn}(x, m)$, even though $\text{cn}(x, m)\text{dn}(x, m)$ is not a solution of these nonlinear equations. Finally, we also obtain superposed solutions of various forms for several coupled nonlinear equations.

Date of resubmission: February 7, 2014

1 Introduction

Nonlinear equations are playing an increasingly important role in several areas of science in general and physics in particular. One of the major problem with these equations is the lack of a superposition principle. In this context it is worth recalling that the linear superposition principle is one of the hallmarks of linear theories which does not hold good in nonlinear theories because of the nonlinear term(s). Thus, even if two solutions are known for a nonlinear theory, their superposition is in general not a solution of that nonlinear theory. The purpose of this paper is to point out a novel kind of superposition which seems to hold good for a large number of nonlinear equations, both discrete and continuum. In particular, there are several nonlinear field equations, discrete as well as continuum [1, 2, 3], which are known to admit periodic solutions in terms of Jacobi elliptic functions (JEF) $\text{cn}(x, m)$ and $\text{dn}(x, m)$, where m denotes the modulus of the elliptic function [4]. Many of these solutions have found wide application in several areas of physics [5, 6, 7]. Our goal is to show, through a large number of examples, that a kind of novel linear superposition seems to hold good in these cases. We might add here that we do not have a rigorous proof for such a superposition but we have examined a large number of examples, which without exception, seem to support this conjecture. In particular, we examine a number of nonlinear equations, both continuum and discrete, both integrable and nonintegrable, which admit periodic solutions in terms of $\text{cn}(x, m)$ and $\text{dn}(x, m)$ functions and show that in all these cases, without exception, $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$ are also exact periodic solutions.

The continuum nonlinear equations that we have studied are the nonlinear Schrödinger equation (NLSE), quadratic-cubic NLSE [8, 9], MKdV [5, 6, 7], mixed KdV-MKdV system, $\lambda\phi^4$ field theory [5, 6, 7], etc. On the other hand, the discrete nonlinear equations that we have examined are the Ablowitz-Ladik equation [10, 11], saturable discrete NLSE [12], discrete MKdV, discrete $\lambda\phi^4$ field theory [13], discrete cubic-quintic model, etc. Amongst these, NLSE, MKdV, Ablowitz-Ladik and discrete MKdV are the integrable models while the rest are not integrable. In addition, we have studied several coupled nonlinear equations, e.g. coupled ϕ^4 [14], coupled NLS-MKdV system, coupled KdV-quadratic NLS, coupled NLS (including the Manakov system [15]), etc. and find that they also admit such superposed solutions.

Further, we also examine a number of continuum field theories like KdV [5, 6, 7], quadratic NLS and

ϕ^3 field theory [16, 17] which admit $\text{dn}^2(x, m)$ as a periodic solution and show that all these models also admit periodic solutions of the form $\text{dn}^2(x, m) \pm \sqrt{m}\text{cn}(x, m)\text{dn}(x, m)$ even though $\text{cn}(x, m)\text{dn}(x, m)$ is not a solution of such models. While this cannot be viewed as a linear superposition of two solutions since $\text{cn}(x, m)\text{dn}(x, m)$ is not a solution of these models, it is rather remarkable that such solutions exist, without exception, in a number of continuum field theory models, both integrable and nonintegrable, that we have examined.

We have also considered several coupled field theories in which while one field admits a periodic solution of the form $\text{dn}^2(x, m)$, the other field either admits a periodic solution of the form $\text{dn}(x, m)$ or $\text{cn}(x, m)$ and in all such cases, without fail, we find that the coupled model also admits periodic solutions of the form $\text{dn}^2(x, m) \pm \sqrt{m}\text{cn}(x, m)\text{dn}(x, m)$ in one field and $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$ type solutions in the other field. We might add here that so far as we are aware of, the solutions obtained in this paper are new and were not known previously. Additionally, the superposed solutions that we have obtained are not connected to the known solutions by any kind of Landen transformation [4].

The paper is organized as follows. In Sec. II we discuss several continuum models which admit $\text{cn}(x, m)$ as well as $\text{dn}(x, m)$ as periodic solutions and show that such models also admit periodic solutions of the form $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$. In Sec. III We discuss a few coupled continuum models in which both the fields are known to admit $\text{cn}(x, m)$ and $\text{dn}(x, m)$ as their exact solution and show that such coupled models also admit superposed solutions of the form $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$ in both the fields. In Sec. IV we discuss several discrete nonlinear equations which are known to admit $\text{cn}(x, m)$ and $\text{dn}(x, m)$ as periodic solutions and show that all of them also admit $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$ as solutions. In Sec. V we discuss a few continuum models which admit $\text{dn}^2(x, m)$ as a periodic solution and show that such models also admit $\text{dn}^2(x, m) \pm \sqrt{m}\text{cn}(x, m)\text{dn}(x, m)$ as periodic solutions even though $\text{cn}(x, m)\text{dn}(x, m)$ is not a solution of these models. In Sec. VI we discuss few coupled continuum models in which both the fields are known to admit dn^2 as their exact solution and show that they also admit solutions of the form $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ in both the fields. In Sec. VII we discuss a few coupled continuum field theories in which one of the field admits $\text{cn}(x, m)$ as well as $\text{dn}(x, m)$ as an exact solution while the other field has $\text{dn}^2(x, m)$ as an exact solution. We show that such models also admit $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$ as solution in the first field and

$\text{dn}^2(x, m) \pm \sqrt{m}\text{cn}(x, m)\text{dn}(x, m)$ as solution in the second field. Some preliminary results have appeared previously [18]. We summarize our main conclusions in Sec. VIII where we also discuss possible reasons why the linear superposition of $\text{dn}(x, m)$ and $\text{cn}(x, m)$ is also a solution of models which admit $\text{dn}(x, m)$ and $\text{cn}(x, m)$ as solutions.

2 $\text{dn} \pm \sqrt{m}\text{cn}$ as Exact Solutions of Continuum Nonlinear Equations

In this section, we discuss six continuum models, all of which admit periodic solutions in terms of Jacobi elliptic functions (JEF) $\text{dn}(x, m)$ as well as $\text{cn}(x, m)$, and show that, in all these cases, $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$ are also exact solutions. Hence forth, for the sake of brevity, we will omit the arguments (x, m) of JEF in the text.

2.1 NLS Equation

We start with the nonlinear Schrödinger (NLS) equation [1, 2]

$$iu_t + u_{xx} + g|u|^2u = 0, \quad (1)$$

which is a well known integrable model. It has found applications in several branches of physics [1, 2]. It is well known that one of the exact moving periodic solution to this equation is

$$u = A\text{dn}[\beta(x - vt + \delta_1), m] \exp[-i(\omega t - kx + \delta)], \quad (2)$$

provided

$$gA^2 = 2\beta^2, \quad \omega = k^2 - (2 - m)\beta^2, \quad v = 2k. \quad (3)$$

Here δ, δ_1 are two arbitrary constants arising due to translational invariance. In fact this is true for all the models discussed in this paper and hence we will not mention about δ, δ_1 any more in this paper.

Another exact solution to NLS Eq. (1) is

$$u = A\sqrt{m}\text{cn}[\beta(x - vt + \delta_1), m] \exp[-i(\omega t - kx + \delta)] \quad (4)$$

provided

$$gA^2 = 2\beta^2, \quad \omega = k^2 - (2m - 1)\beta^2, \quad v = 2k. \quad (5)$$

Remarkably, even a linear superposition of the two, i.e.

$$u = \left(\frac{A}{2} \text{dn}[\beta(x - vt + \delta_1), m] + \frac{B}{2} \sqrt{m} \text{cn}[\beta(x - vt + \delta_1), m] \right) \exp[-i(\omega t - kx + \delta)], \quad (6)$$

is an exact solution to the NLS Eq. (1) provided

$$B = \pm A, \quad gA^2 = 2\beta^2, \quad \omega - k^2 = -(1/2)(1 + m)\beta^2, \quad v = 2k. \quad (7)$$

It is worth noting that the frequency ω associated with the three solutions [i.e. cn , dn , and $(\text{dn} \pm \sqrt{m} \text{cn})$] is different except at $m = 1$. We thus have two new periodic solutions of NLSE depending on whether $B = A$ or $B = -A$. Few remarks are in order here which are in fact valid for all the solutions (both continuum and discrete) discussed in Secs. II, III and IV. Therefore, we shall not repeat these remarks while discussing various solutions in these three sections.

1. All the models in these three sections admit cn , dn as well as $\text{dn} \pm \sqrt{m} \text{cn}$ as solutions. It is insightful to note that both the solutions $\text{dn} + \sqrt{m} \text{cn}$ and $\text{dn} - \sqrt{m} \text{cn}$ exist for the same values of the parameters. It will be interesting to know the region of stability of these two solutions vis a vis those of dn and cn solutions.
2. In the limit $m = 1$, all three solutions dn , cn as well as $\text{dn} + \sqrt{m} \text{cn}$ go over to the well known pulse (i.e. sech) solution while $\text{dn} - \sqrt{m} \text{cn}$ goes over to the vacuum solution.
3. In all the continuum models discussed in this section, the factors of $2 - m$ or $2m - 1$ which appear in the dn and cn solutions, get replaced by the factor of $(1 + m)/2$ in the $\text{dn} \pm \sqrt{m} \text{cn}$ solutions.
4. On the other hand, in the discrete models discussed in Sec. IV, the factors of $\text{dn}(\beta, m)$ or $\text{cn}(\beta, m)$ appearing in dn and cn solutions, get replaced by the factor of $[\text{dn}(\beta, m) + \text{cn}(\beta, m)]/2$ in the $\text{dn} \pm \sqrt{m} \text{cn}$ solutions.

2.2 MKdV Equation

We now show that the celebrated MKdV equation [5, 6]

$$u_t + u_{xxx} + gu^2u_x = 0, \quad (8)$$

which is a well known integrable equation and has found applications in several areas [5, 6], also admits such superposed solutions.

It is well known that one of the exact solution to the MKdV Eq. (8) is

$$u = A \operatorname{dn}[\beta(x - vt + \delta_1), m], \quad (9)$$

provided

$$gA^2 = 6\beta^2, \quad v = (2 - m)\beta^2. \quad (10)$$

Similarly, another exact solution to the MKdV Eq. (8) is

$$u = A\sqrt{m}\operatorname{cn}[\beta(x - vt + \delta_1), m], \quad (11)$$

provided

$$gA^2 = 6\beta^2, \quad v = (2m - 1)\beta^2. \quad (12)$$

Remarkably, even a linear superposition of the two, i.e.

$$u = \frac{A}{2}\operatorname{dn}[\beta(x - vt + \delta_1), m] + \frac{B}{2}\sqrt{m}\operatorname{cn}[\beta(x - vt + \delta_1), m], \quad (13)$$

is an exact solution to the MKdV Eq. (8) provided

$$B = \pm A, \quad gA^2 = 6\beta^2, \quad v = (1/2)(1 + m)\beta^2. \quad (14)$$

Note that the velocity of the solutions dn , cn and $\operatorname{dn} \pm \sqrt{m}\operatorname{cn}$ is different except at $m = 1$.

2.3 ϕ^2 - ϕ^4 Model

We now show that the ϕ^2 - ϕ^4 field equation [7]

$$\phi_{xx} = a\phi + b\phi^3, \quad (15)$$

also admits such superposed solutions.

It is well known that one of the exact solution to the field Eq. (15) is

$$\phi = A \operatorname{dn}[\beta(x - vt + \delta_1), m], \quad (16)$$

provided

$$bA^2 = -2\beta^2, \quad a = (2 - m)\beta^2. \quad (17)$$

This implies that $a > 0, b < 0$.

Another exact solution to the field Eq. (15) is

$$\phi = A\sqrt{m}\text{cn}[\beta(x - vt + \delta_1), m], \quad (18)$$

provided

$$bA^2 = -2\beta^2, \quad a = (2m - 1)\beta^2. \quad (19)$$

Thus this solution is valid if $b < 0$ while $a > (<) 0$ depending on whether $m > (<) 1/2$. Remarkably, even a linear superposition of the two, i.e.

$$\phi = \frac{A}{2}\text{dn}[\beta(x - vt + \delta_1), m] + \frac{B}{2}\sqrt{m}\text{cn}[\beta(x - vt + \delta_1), m], \quad (20)$$

is an exact solution to the field Eq. (15) provided

$$B = \pm A, \quad bA^2 = -2\beta^2, \quad a = (1/2)(1 + m)\beta^2. \quad (21)$$

Unlike the cn (but like the dn solution), these solutions exist only if $a > 0$. Note that the value of the width parameter β is different for the three solutions.

2.4 ϕ^2 - ϕ^3 - ϕ^4 case

The asymmetric double well potential arises in field theory [19] as well as in the context of certain first order phase transitions [20]. The ϕ^2 - ϕ^3 - ϕ^4 field equation is [19, 20]

$$\phi_{xx} = a\phi + b\phi^2 + c\phi^3. \quad (22)$$

It admits the periodic solution

$$u = A + B\text{dn}[\beta(x - vt + \delta_1), m], \quad (23)$$

provided

$$A = -\frac{b}{3c}, \quad cB^2 = -2\beta^2, \quad a = -2(2 - m)\beta^2, \quad 2b^2 = 9|a||c|. \quad (24)$$

This implies that $a < 0, b > 0, c < 0$.

Another exact solution to the field Eq. (22) is

$$u = A + B\sqrt{m}\text{cn}[\beta(x - vt + \delta_1), m], \quad (25)$$

provided

$$A = -\frac{b}{3c}, \quad cB^2 = -2\beta^2, \quad a = -2(2m - 1)\beta^2, \quad 2b^2 = 9|a||c|. \quad (26)$$

Thus this solution exists only if $b > 0, c < 0$ while $a < (> 0)$ or $= 0$ depending on whether $m > (<) 1/2$ or $= 0$.

Remarkably, even a linear superposition of the two, i.e.

$$u = A + \frac{B}{2}\text{dn}[\beta(x - vt + \delta_1), m] + \frac{D}{2}\sqrt{m}\text{cn}[\beta(x - vt + \delta_1), m], \quad (27)$$

is an exact solution to the field Eq. (22) provided

$$D = \pm B, \quad A = -\frac{b}{3c}, \quad cB^2 = -2\beta^2, \quad a = -(m + 1)\beta^2, \quad 2b^2 = 9|a||c|. \quad (28)$$

Note that as in the dn case, such solutions exist only if $a, c < 0$ while $b > 0$.

2.5 Mixed KdV-MKdV system

The field equations of the mixed KdV-MKdV system are given by

$$u_t + \delta u_{xxx} + \alpha u^2 u_x + \gamma u u_x = 0. \quad (29)$$

It is easy to show that one of the exact periodic solution to Eq. (29) is

$$u(x, t) = A + B\text{dn}[\beta(x - vt + \delta_1), m], \quad (30)$$

provided

$$A = -\frac{\gamma}{2\alpha}, \quad B^2 = \frac{6\delta\beta^2}{\alpha}, \quad (31)$$

$$v = (2 - m)\delta\beta^2 - \frac{\gamma^2}{4\alpha}. \quad (32)$$

Another periodic solution to Eq. (29) is

$$u(x, t) = A + B\sqrt{m}\text{cn}[\beta(x - vt + \delta_1), m], \quad (33)$$

provided relations (31) are satisfied while the velocity v is given by

$$v = (2m - 1)\delta\beta^2 - \frac{\gamma^2}{4\alpha}. \quad (34)$$

Remarkably, even a superposition of the two solutions (30) and (33) is also an exact solution but with velocity v which is different than that given by either Eq. (32) or (34). In particular, it is easy to show that

$$u(x, t) = A + \frac{B}{2}\text{dn}[\beta(x - vt + \delta_1), m] + \frac{D}{2}\sqrt{m}\text{cn}[\beta(x - vt + \delta), m], \quad (35)$$

is an exact solution to the field Eq. (29) provided relations (31) are satisfied and further

$$D = \pm B, \quad v = \frac{(1 + m)}{2}\delta\beta^2 - \frac{\gamma^2}{4\alpha}. \quad (36)$$

Notice that the velocities of the three solutions dn , cn and $\text{dn} \pm \sqrt{m}\text{cn}$ are different.

2.6 Mixed Quadratic-Cubic NLS Equation

Let us consider a mixed quadratic-cubic NLS equation [8, 9] given by

$$iu_t + u_{xx} + g_1|u|u + g_2|u|^2u = 0. \quad (37)$$

One of the exact moving periodic solution to this equation is

$$u = (A\text{dn}[\beta(x - vt + \delta_1), m] + B)\exp[-i(\omega t - kx + \delta)], \quad (38)$$

provided

$$g_1 = -3Bg_2, \quad g_2A^2 = 2\beta^2, \quad g_2B^2 = (2 - m)\beta^2, \quad \omega = k^2 + 2(2 - m)\beta^2, \quad v = 2k. \quad (39)$$

Similarly, another exact solution to Eq. (37) is

$$u = (A\sqrt{m}\text{cn}[\beta(x - vt + \delta_1), m] + B)\exp[-i(\omega t - kx + \delta)] \quad (40)$$

provided

$$g_1 = -3Bg_2, \quad g_2A^2 = 2\beta^2, \quad g_2B^2 = (2m - 1)\beta^2, \quad \omega = k^2 + 2(2m - 1)\beta^2, \quad v = 2k. \quad (41)$$

Note that this solution exists only if $m > 1/2$.

Remarkably, a linear superposition of the two, i.e.

$$u = \left(\frac{A}{2} \text{dn}[\beta(x - vt + \delta_1), m] + \frac{D}{2} \sqrt{m} \text{cn}[\beta(x - vt + \delta_1), m] + B \right) \exp[-i(\omega t - kx + \delta)], \quad (42)$$

is an exact solution to the field Eq. (37) provided

$$\begin{aligned} B &= \pm A, \quad g_1 = -3Bg_2, \quad g_2 A^2 = 2\beta^2, \\ g_2 B^2 &= \frac{1+m}{2} \beta^2, \quad \omega = k^2 + (1/2)(1+m)\beta^2, \quad v = 2k. \end{aligned} \quad (43)$$

Note that even though the cn solution is only valid if $m > 1/2$, the superposed solution of cn and dn is in fact valid over the entire range of m values, i.e. $0 < m \leq 1$. Further, the frequency ω of the three solutions (i.e. cn, dn, and $(\text{dn} \pm \sqrt{m} \text{cn})$) is different except at $m = 1$.

3 Coupled Continuum Field Theories with $\text{dn} \pm \sqrt{m} \text{cn}$ Solutions in Both Fields

We now show that several coupled field theory models (which admit solutions in terms of cn and dn in both the fields) also admit $\text{dn} \pm \sqrt{m} \text{cn}$ as solutions in both the fields. As an illustration, we discuss three examples of coupled continuum field theories which admit such solutions.

3.1 Coupled ϕ^4 Field Theories

Some time ago, we had considered the coupled ϕ^4 field theories with field equations given by [14]

$$\begin{aligned} \phi_{xx} &= 2\alpha_1 \phi + 4\beta_1 \phi^3 + 2\gamma \phi \psi^2, \\ \psi_{xx} &= 2\alpha_2 \psi + 4\beta_2 \psi^3 + 2\gamma \psi \phi^2. \end{aligned} \quad (44)$$

The four well known periodic solutions to these coupled equations are $\phi = A \text{dn}$ or $\phi = A \sqrt{m} \text{cn}$ and $\psi = B \text{dn}$ or $\psi = B \sqrt{m} \text{cn}$. For illustration, we just discuss one of the known solutions. The details of the other three solutions can be found in [14]. For example, one of the solutions is given by

$$\phi = A \text{dn}[\beta(x + \delta_1), m], \quad \psi = D \sqrt{m} \text{cn}[\beta(x + \delta_1), m], \quad (45)$$

provided

$$\beta^2 = -2\beta_1 A^2 - \gamma D^2 = -2\beta_2 D^2 - \gamma A^2, \quad (46)$$

$$\alpha_1 = \frac{(2-m)\beta^2 A^2}{2} + \gamma(1-m)D^2, \quad \alpha_2 = \frac{(2m-1)\beta^2 D^2}{2} - \gamma(1-m)A^2. \quad (47)$$

On solving Eq. (46) we have

$$A^2 = \frac{2|\beta_2| - |\gamma|}{4\beta_1\beta_2 - \gamma^2}, \quad D^2 = \frac{2|\beta_1| - |\gamma|}{4\beta_1\beta_2 - \gamma^2}. \quad (48)$$

In the special case when $\gamma = 2\beta_1 = 2\beta_2 < 0$, instead of the relations (48), A, D only satisfy the constraint

$$\beta^2 = |\gamma|(A^2 + D^2). \quad (49)$$

We now show that even a linear superposition of dn and cn (in both the fields) is an exact solution of Eqs. (44). In particular, it is easily checked that

$$\phi = \frac{A}{2} \text{dn}[\beta(x + \delta_1), m] + \frac{B}{2} \sqrt{m} \text{cn}[\beta(x + \delta_1), m], \quad (50)$$

$$\psi = \frac{D}{2} \text{dn}[\beta(x + \delta_1), m] + \frac{E}{2} \sqrt{m} \text{cn}[\beta(x + \delta_1), m], \quad (51)$$

is an exact solution to the coupled Eqs. (44) provided

$$B = \pm A, \quad E = \pm D, \quad \alpha_1 = \alpha_2 = \frac{(1+m)\beta^2}{4}, \quad (52)$$

while A, D satisfy Eq. (46) and hence relations (48) or constraint (49). Note that the signs of $D = \pm A$ and $E = \pm B$ are correlated.

3.2 Coupled NLS-MKdV Model

We now consider a coupled NLS-MKdV system with the field equations given by

$$\begin{aligned} iu_t + u_{xx} + g|u|^2 u + \alpha uv^2 &= 0, \\ v_t + v_{xxx} + 6v^2 v_x + \gamma v(|u|^2)_x &= 0, \end{aligned} \quad (53)$$

where u and v are the NLS and the MKdV fields, respectively. As we remarked in Ref. [18], these coupled equations admit four periodic solutions with u being either cn or dn (multiplied by an exponential) and

similarly v can be either cn or dn. However, we only discuss one of the four solutions here, given by

$$\begin{aligned} u(x, t) &= A \exp[-i(\omega t - kx + \delta)] \text{dn}[\beta(x - ct + \delta_1), m], \\ v(x, t) &= B \sqrt{m} \text{cn}[\beta(x - ct + \delta_1), m], \end{aligned} \quad (54)$$

provided

$$c = 2k = (2m - 1)\beta^2, \quad \omega = k^2 - (2 - m)\beta^2 + (1 - m)\alpha B^2, \quad (55)$$

$$gA^2 + \alpha B^2 = 2\beta^2, \quad \gamma A^2 + 3B^2 = 3\beta^2. \quad (56)$$

On solving Eqs. (56), we obtain

$$A^2 = \frac{3(\alpha - 2)\beta^2}{\alpha\gamma - 3g}, \quad B^2 = \frac{(2\gamma - 3g)\beta^2}{(\alpha\gamma - 3g)}. \quad (57)$$

In the special case when $\gamma = (3/2)g, \alpha = 2$, A, B remain undetermined, and instead of Eqs. (57) A, B only satisfy the constraint

$$\gamma A^2 + 3B^2 = 3\beta^2. \quad (58)$$

Remarkably, even a linear superposition

$$\begin{aligned} u(x, t) &= \frac{1}{2} \exp[-i(\omega t - kx + \delta)] \left(A \text{dn}[\gamma(x - ct + \delta_1), m] \right. \\ &\quad \left. + D \sqrt{m} \text{cn}[\gamma(x - ct + \delta_1), m] \right), \end{aligned} \quad (59)$$

and

$$v(x, t) = \frac{1}{2} \left(B \text{dn}[\gamma(x - ct + \delta_1), m] + F \sqrt{m} \text{cn}[\gamma(x - ct + \delta_1), m] \right), \quad (60)$$

is an exact solution of Eqs. (53) provided

$$c = 2k = (1 + m)\gamma^2/2, \quad \omega = k(k - 2), \quad D = \pm A, \quad F = \pm B, \quad (61)$$

while A, B are either given by Eqs. (57) or are related by the constraint (58). Note that the signs of $D = \pm A$ and $F = \pm B$ are correlated.

3.3 Coupled NLS Model

Let us consider the following coupled NLS field equations

$$\begin{aligned} iu_t + u_{xx} + (a|u|^2 + b|v|^2)u &= 0, \\ v_t + v_{xx} + (f|u|^2 + e|v|^2)v &= 0, \end{aligned} \quad (62)$$

where u and v are the two coupled NLS fields. Note that in the special case when $a = f = b = e$ this system reduces to the Manakov system which is a well known integrable system [15]. Remarkably, even when $a = f = -b = -e$, this is an integrable system [21, 22, 23] which we shall call as MZS (Mikhailov-Zakharov-Schulman) system. We shall however discuss the exact periodic solutions of this coupled system when the coefficients a, b, f, e are arbitrary but real.

The coupled equations (62) admit four solutions with either cn or dn in u as well as v fields and several other solutions in terms of Lamé polynomials of order 1 and 2. Here, as an illustration, we only discuss one such solution and then show that these coupled equations also admit solutions in terms of a linear superposition of dn and cn in both the fields.

It is easily checked that

$$u(x, t) = A \exp[-i(\omega_1 t - k_1 x + \delta_1)] \text{dn}[\beta(x - ct + \delta), m], \quad (63)$$

and

$$v(x, t) = B\sqrt{m} \exp[-i(\omega_2 t - k_2 x + \delta_2)] \text{cn}[\beta(x - ct + \delta), m], \quad (64)$$

is an exact solution to the coupled field equations (62) provided

$$aA^2 + bB^2 = 2\beta^2, \quad fA^2 + eB^2 = 2\beta^2, \quad (65)$$

and further

$$k_1 = k_2, \quad c = 2k_1, \quad \omega_1 = k_1^2 - (2 - m)\beta^2 - (1 - m)bB^2, \quad \omega_2 = k_1^2 - (2m - 1)\beta^2 - (1 - m)fA^2. \quad (66)$$

On solving Eqs. (65) we find that so long as $bf \neq ae$, A, B are given by

$$A^2 = \frac{2\beta^2(b - e)}{bf - ae}, \quad B^2 = \frac{2\beta^2(f - a)}{bf - ae}. \quad (67)$$

Few remarks are in order at this stage.

1. In case $ae = bf$, then along with Eqs. (65) this implies that $b = e$ and $a = f$. In that case instead of the relations (67), we only have the constraint

$$aA^2 + bB^2 = 2\beta^2. \quad (68)$$

2. In the Manakov case, $a = b = e = f$ and the constraint (68) becomes $a(A^2 + B^2) = 2\beta^2$. On the other hand in the MZS case when $a = f = -e = -b$, the constraint becomes $a(A^2 - B^2) = 2\beta^2$.

Remarkably, it turns out that even a linear superposition of dn and cn is a solution to the coupled Eqs. (62). In particular,

$$u(x, t) = \frac{1}{2} \exp[-i(\omega_1 t - k_1 x + \delta_1)] \left(A \operatorname{dn}[\beta(x - ct + \delta), m] + \sqrt{m} D \operatorname{cn}[\beta(x - ct + \delta), m] \right), \quad (69)$$

and

$$v(x, t) = \frac{1}{2} \exp[-i(\omega_2 t - k_2 x + \delta_2)] \left(B \operatorname{dn}[\beta(x - ct + \delta), m] + \sqrt{m} E \operatorname{cn}[\beta(x - ct + \delta), m] \right), \quad (70)$$

is an exact solution to the coupled field equations (62) provided Eqs. (65) are satisfied and further

$$k_1 = k_2, \quad c = 2k_1, \quad D = \pm A, \quad E = \pm B, \quad \omega_1 = \omega_2 = k_1^2 - \frac{1}{2}(1 + m)\beta^2. \quad (71)$$

Note that the signs of $D = \pm A$ and $E = \pm B$ are correlated. Further, all the remarks made after the previous solution are also valid for this case.

4 Discrete Nonlinear Equations

We now discuss five examples of discrete nonlinear equations all of which are known to admit dn and cn as periodic solutions. We show that all these models also admit periodic solutions of the form $\operatorname{dn} \pm \sqrt{m} \operatorname{cn}$.

4.1 Ablowitz-Ladik Model

It is well known that the celebrated Ablowitz-Ladik (AL) model [10, 11], which is an integrable model, admits moving dn and cn periodic solutions [24]. We now show that the same model also admits linearly superposed moving periodic solutions.

We start from the AL equation

$$i \frac{du_n}{dt} + u_{n+1} + u_{n-1} + |u_n|^2 [u_{n+1} + u_{n-1}] = 0. \quad (72)$$

An exact moving periodic solution to Eq. (72) is known to be

$$u_n = \text{Adn}[\beta(n - vt + \delta_1), m] e^{-i(\omega t - kn + \delta)}, \quad (73)$$

provided

$$\omega = -\frac{2 \cos(k) \text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}, \quad \beta v = \frac{2 \sin(k)}{\text{cs}(\beta, m)}, \quad gA^2 \text{cs}^2(\beta, m) = 1. \quad (74)$$

Another exact moving soliton solution to Eq. (72) is

$$u_n = A \sqrt{m} \text{cn}[\beta(n - vt + \delta_1), m] e^{-i(\omega t - kn + \delta)}, \quad (75)$$

provided

$$\omega = -\frac{2 \cos(k) \text{cn}(\beta, m)}{\text{dn}^2(\beta, m)}, \quad \beta v = \frac{2 \sin(k)}{\text{ds}(\beta, m)}, \quad gA^2 \text{ds}^2(\beta, m) = 1. \quad (76)$$

Remarkably, even a linear superposition of the dn and cn solutions is also an exact solution to Eq. (72).

In particular, it is easy to show that

$$u_n = \left(\frac{A}{2} \text{dn}[\beta(n - vt + \delta_1), m] + \frac{B}{2} \sqrt{m} \text{cn}[\beta(n - vt + \delta_1), m] \right) e^{-i(\omega t - kn + \delta)}, \quad (77)$$

is also an exact solution to Eq. (72) provided

$$\begin{aligned} B &= \pm A, \quad \omega = -\frac{4 \cos(k)}{(\text{cn}(\beta, m) + \text{dn}(\beta, m))}, \\ \beta v &= \frac{4 \sin(k)}{[\text{cs}(\beta, m) + \text{ds}(\beta, m)]}, \quad gA^2 [\text{cs}(\beta, m) + \text{ds}(\beta, m)]^2 = 4. \end{aligned} \quad (78)$$

As remarked earlier (in Sec. II), observe that if we replace $\text{dn}(\beta, m)$ and $\text{cn}(\beta, m)$ by $[\text{dn}(\beta, m) + \text{cn}(\beta, m)]/2$ in relations (74) and (76), we recover relations (78). Exactly the same observation is also valid in the case of the next four discrete solutions that we discuss below.

4.2 Saturable DNLS Equation

Let us consider the saturable discrete nonlinear Schrödinger (DNLS) equation

$$i \frac{du_n}{dt} + [u_{n+1} + u_{n-1} - 2u_n] + \frac{\nu |u_n|^2 u_n}{1 + |u_n|^2} = 0. \quad (79)$$

It is worth reminding that this equation has received great attention in the context of optical pulse propagation in various doped fibers [25]. It may also be relevant for the description of arrays of optical waveguides with nonpolynomial nonlinearities and Bose-Einstein condensates [26].

There are two well known periodic solutions to this equation [12]. The first one is

$$u_n = A \operatorname{dn}[\beta(n + \delta_1), m] e^{-i(\omega t + \delta)}, \quad (80)$$

provided

$$A^2 \operatorname{cs}^2(\beta, m) = 1, \quad \omega = 2 - \nu = 2 \left[1 - \frac{\operatorname{cn}(\beta, m)}{\operatorname{dn}^2(\beta, m)} \right]. \quad (81)$$

The other solution is

$$u_n = A \sqrt{m} \operatorname{cn}[\beta(n + \delta_1), m] e^{-i(\omega t + \delta)}, \quad (82)$$

provided

$$A^2 \operatorname{ds}^2(\beta, m) = 1, \quad \omega = 2 - \nu = 2 \left[1 - \frac{\operatorname{dn}(\beta, m)}{\operatorname{cn}^2(\beta, m)} \right], \quad (83)$$

where $\operatorname{cs}(\beta, m) = \operatorname{cn}(\beta, m)/\operatorname{sn}(\beta, m)$ and $\operatorname{ds}(\beta, m) = \operatorname{dn}(\beta, m)/\operatorname{sn}(\beta, m)$.

Remarkably, even a linear superposition of the two is also an exact periodic solution to the saturable DNLS Eq. (79), i.e. it is easy to show that

$$u_n = \left(\frac{A}{2} \operatorname{dn}[\beta(n + \delta_1), m] + \frac{B}{2} \sqrt{m} \operatorname{cn}[\beta(n + \delta_1), m] \right) e^{-i(\omega t + \delta)}, \quad (84)$$

is also an exact solution to Eq. (79) provided

$$B = \pm A, \quad A^2 [\operatorname{cs}(\beta, m) + \operatorname{ds}(\beta, m)]^2 = 4, \quad \omega = 2 - \nu = 2 \left[1 - \frac{2}{\operatorname{dn}(\beta, m) + \operatorname{cn}(\beta, m)} \right]. \quad (85)$$

4.3 Discrete $\lambda\phi^4$

Consider the discrete $\lambda\phi^4$ field equation

$$\frac{1}{h^2} [\phi_{n+1} + \phi_{n-1} - 2\phi_n] + \lambda\phi_n - \frac{\lambda}{2} \phi_n^2 [\phi_{n+1} + \phi_{n-1}] = 0, \quad (86)$$

which is quite similar to the stationary version of the Ablowitz-Ladik equation, Eq. (72), except that the field ϕ_n is real in the present case. It is well known that Eq. (86) admits the periodic solution [27]

$$\phi_n = A \operatorname{dn}[\beta(n + \delta_1), m], \quad (87)$$

provided

$$\frac{1}{h^2} = -\frac{\lambda A^2}{2} \text{cs}^2(\beta, m), \quad \lambda - \frac{2}{h^2} = \lambda A^2 \text{ds}(\beta, m) \text{ns}(\beta, m). \quad (88)$$

$$\Lambda = \lambda h^2 < 0, \quad A^2 = \frac{2}{|\Lambda| \text{cs}^2(\beta, m)}, \quad \Lambda = 2 \left[1 - \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} \right]. \quad (89)$$

Another known periodic solution to Eq. (86) is

$$\phi_n = A \sqrt{m} \text{cn}[\beta(n + \delta_1), m], \quad (90)$$

provided

$$\frac{1}{h^2} = -\frac{\lambda A^2}{2} \text{ds}^2(\beta, m), \quad \lambda - \frac{2}{h^2} = \lambda A^2 \text{cs}(\beta, m) \text{ns}(\beta, m). \quad (91)$$

$$\Lambda < 0, \quad A^2 = \frac{2}{|\Lambda| \text{ds}^2(\beta, m)}, \quad \Lambda = 2 \left[1 - \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)} \right]. \quad (92)$$

We now show that the same model (86) also admits superposed solution of cn and dn. In particular, it is easy to check that the model admits an exact solution

$$\phi_n = \frac{A}{2} \text{dn}[\beta(n + \delta_1), m] + \frac{B}{2} \sqrt{m} \text{cn}[\beta(n + \delta_1), m], \quad (93)$$

provided

$$B = \pm A, \quad \Lambda < 0, \quad A^2 = \frac{8}{|\Lambda| [\text{cs}(\beta, m) + \text{ds}(\beta, m)]^2}, \quad \Lambda = 2 \left[1 - \frac{2}{\text{cn}(\beta, m) \text{dn}(\beta, m)} \right]. \quad (94)$$

4.4 Discrete Cubic-Quintic Model

There is a relation between the continuum generalized NLS and the cubic-quintic NLS [28]. In the present discrete case, the discrete field equations are

$$i \frac{du_n}{dt} + [u_{n+1} + u_{n-1}] + g_1 |u_n|^4 [u_{n+1} + u_{n-1}] + g_2 |u_n|^2 u_n = 0. \quad (95)$$

An exact periodic solution to Eq. (95) is known to be [13]

$$u_n = A \text{dn}[\beta(n + \delta_1), m] e^{-i(\omega t + \delta)}, \quad (96)$$

provided

$$g_1 < 0, \quad \omega = -2 \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}, \quad |g_1| A^4 \text{cs}^4(\beta, m) = 1, \quad \frac{g_2}{2\sqrt{|g_1|}} = \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}. \quad (97)$$

Another exact periodic solution to Eq. (95) is

$$u_n = A\sqrt{m}\text{cn}[\beta(n + \delta_1), m]e^{-i(\omega t + \delta)}, \quad (98)$$

provided

$$g_1 < 0, \quad \omega = -2\frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)}, \quad |g_1|A^4\text{ds}^4(\beta, m) = 1, \quad \frac{g_2}{2\sqrt{|g_1|}} = \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)}. \quad (99)$$

Remarkably, even a linear superposition of the above two is also an exact solution to the Eq. (95). In particular, it is easy to show that

$$u_n = \left(\frac{A}{2}\text{dn}[\beta(n + \delta_1), m] + \frac{B}{2}\sqrt{m}\text{cn}[\beta(n + \delta_1), m] \right) e^{-i(\omega t + \delta)}, \quad (100)$$

is also a periodic solution to Eq. (95) provided

$$\begin{aligned} g_1 < 0, \quad \omega &= -\frac{4}{(\text{cn}(\beta, m) + \text{dn}(\beta, m))}, \quad |g_1|A^4[\text{ds}(\beta, m) + \text{cs}(\beta, m)]^4 = 16, \\ B = \pm A, \quad \frac{g_2}{2\sqrt{|g_1|}} &= \frac{2}{\text{dn}(\beta, m) + \text{cn}(\beta, m)}. \end{aligned} \quad (101)$$

4.5 Discrete MKdV Model

The discrete MKdV equation is known to be an integrable equation [10] and is given by

$$\frac{du_n}{dt} + \alpha[u_{n+1} + u_{n-1}] + u_n^2[u_{n+1} - u_{n-1}] = 0, \quad (102)$$

where $u_n(t)$ is a real field. We now show that not only the real but even the complex discrete MKdV equation

$$\frac{du_n}{dt} + \alpha[u_{n+1} + u_{n-1}] + |u_n|^2[u_{n+1} - u_{n-1}] = 0, \quad (103)$$

has such superposed solutions.

An exact moving periodic solution to Eq. (103) is

$$u_n = A\text{dn}[\beta(n - vt + \delta_1), m]e^{-i(\omega t - kn + \delta)}, \quad (104)$$

provided

$$\omega = \frac{2\alpha \sin(k)\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}, \quad \beta v = \frac{2\alpha \cos(k)}{\text{cs}(\beta, m)}, \quad A^2\text{cs}^2(\beta, m) = \alpha > 0. \quad (105)$$

It may be noted that in the limit $k = 0$, $\omega = 0$, the solution (104) is an exact solution to the real, discrete MKdV Eq. (102).

Another exact moving soliton solution to Eq. (103) is

$$u_n = A\sqrt{m}\text{cn}[\beta(n - vt + \delta_1), m]e^{-i(\omega t - kn + \delta)}, \quad (106)$$

provided

$$\omega = \frac{2\alpha \sin(k)\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)}, \quad \beta v = \frac{2\alpha \cos(k)}{\text{ds}(\beta, m)}, \quad A^2 \text{ds}^2(\beta, m) = \alpha > 0. \quad (107)$$

In the limit $k = \omega = 0$, it reduces to an exact solution of the real, discrete MKdV Eq. (102).

Remarkably, even a linear superposition of the dn and cn solutions is also an exact solution to Eq. (103). In particular, it is easy to show that

$$u_n = \left(\frac{A}{2} \text{dn}[\beta(n - vt + \delta_1), m] + \frac{B}{2} \sqrt{m} \text{cn}[\beta(n - vt + \delta_1), m] \right) e^{-i(\omega t - kn + \delta)}, \quad (108)$$

is also an exact solution to Eq. (103) provided

$$B = \pm A, \quad \omega = \frac{4\alpha \sin(k)}{[(\text{cn}(\beta, m) + \text{dn}(\beta, m))]},$$

$$\beta v = \frac{4\alpha \cos(k)}{[\text{cs}(\beta, m) + \text{ds}(\beta, m)]}, \quad A^2 [\text{cs}(\beta, m) + \text{ds}(\beta, m)]^2 = 4\alpha > 0. \quad (109)$$

In the limit $k = \omega = 0$, it reduces to an exact solution of the real, discrete MKdV Eq. (102).

While deriving the various solutions in this section, several not so well known identities for the Jacobi Elliptic Functions (JEF) have been used which have been obtained by one of us a few years ago [29]; they are given in the Appendix.

5 $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ superposed Solutions

We now discuss three examples of continuum field theories which admit dn^2 as a solution and show that all three models also admit solutions of the form $\text{dn}^2 \pm \sqrt{m}\text{cndn}$, even though cndn is not a solution of any of these models.

5.1 KdV Equation

We first discuss the celebrated KdV equation

$$u_t + u_{xxx} + guu_x = 0, \quad (110)$$

which is a well known integrable equation having application in several areas including shallow water waves [5, 6]. It is well known that it admits periodic soliton solution of the form dn^2 . We now show that it also admits superposed solutions of the form $\text{dn}^2 \pm \sqrt{m}\text{cn}\text{dn}$ even though cndn is not a solution of the KdV equation.

It is well known that one of the exact solution to the KdV Eq. (110) is

$$u = A\text{dn}^2[\beta(x - vt + \delta_1), m], \quad (111)$$

provided

$$gA = 12\beta^2, \quad v = 4(2 - m)\beta^2. \quad (112)$$

Remarkably, even

$$u = \frac{1}{2} \left(A\text{dn}^2[\beta(x - vt + \delta_1), m] + B\sqrt{m}\text{cn}[\beta(x - vt + \delta_1), m]\text{dn}[\beta(x - vt + \delta_1), m] \right), \quad (113)$$

is an exact solution of the KdV Eq. (110) provided

$$B = \pm A, \quad gA = 12\beta^2, \quad v = (5 - m)\beta^2. \quad (114)$$

It is worth repeating once again that cndn is *not* an exact solution to the KdV Eq. (110). We thus have two new periodic solutions of KdV Eq. (110) depending on whether $B = A$ or $B = -A$.

Several remarks are in order here which are in fact valid for all the solutions of the form $\text{dn}^2 \pm \sqrt{m}\text{cn}\text{dn}$ which we discuss in this and section and Sec. VI. Therefore, we shall not repeat these remarks while discussing various solutions in these sections.

1. All the models discussed in this and the Sec. VI admit dn^2 as well as $\text{dn}^2 \pm \sqrt{m}\text{cn}\text{dn}$ as solutions even though $\sqrt{m}\text{cn}\text{dn}$ is not a solution of any of these models.

2. In the limit $m = 1$, the two solutions dn^2 and $\text{dn}^2 + \sqrt{m}\text{cn}\text{dn}$ go over to the well known pulse (i.e. sech^2) solution while $\text{dn}^2 - \sqrt{m}\text{cn}\text{dn}$ solution goes over to the vacuum solution.
3. In all the cases discussed in this and the next section, the factors of $2-m$ and $1-m+m^2$ which appear in the dn^2 solution, get replaced by the factors of $(5-m)/4$ and $\sqrt{1+14m+m^2}/4$, respectively, in the $\text{dn}^2 \pm \sqrt{m}\text{cn}\text{dn}$ solutions.

5.2 Quadratic NLS Equation

We show that the quadratic NLS equation given by

$$iu_t + u_{xx} + g|u|u = 0, \quad (115)$$

not only admits $\text{dn}^2 + D$ as a solution but it also admits the superposed solution of the form $\text{dn}^2 \pm \sqrt{m}\text{cn}\text{dn} + D$.

It is easily checked that one of the exact solution to Eq. (115) is

$$u = \left(A \text{dn}^2[\beta(x - vt + \delta_1), m] + D \right) \exp[-i(\omega t - kx + \delta)], \quad (116)$$

provided

$$gA = 6\beta^2, \quad \omega = k^2 + 4(2-m)\beta^2 + 2gD, \quad v = 2k, \quad (117)$$

and

$$gD = -2[(2-m) \pm \sqrt{1-m+m^2}]\beta^2. \quad (118)$$

Remarkably, even

$$u = \left(D + \frac{A}{2} \text{dn}^2[\beta(x - vt + \delta_1), m] + \frac{B}{2} \sqrt{m} \text{cn}[\beta(x - vt + \delta_1), m] \text{dn}[\beta(x - vt + \delta_1), m] \right) \exp[-i(\omega t - kx + \delta)], \quad (119)$$

is an exact solution to Eq. (115) provided

$$v = 2k, \quad B = \pm A, \quad gA = 6\beta^2, \quad \omega = k^2 + (5-m)\beta^2 + 2gD, \quad (120)$$

and further

$$2gD = -[(5-m) \pm \sqrt{1+14m+m^2}]\beta^2. \quad (121)$$

Again note that cndn is not an exact solution to Eq. (115).

5.3 ϕ^3 Field Theory

This field theory arises in the context of third order phase transitions [16] and is also relevant to tachyon condensation [17]. It is well known that the field equation for the $\phi^2 - \phi^3$ field theory given by

$$\phi_{xx} = a\phi + b\phi^2, \quad (122)$$

admits an exact periodic solution

$$\phi = A \operatorname{dn}^2[\beta(x+c), m] + B, \quad (123)$$

provided

$$A = -\frac{3a}{2b\sqrt{1-m+m^2}}, \quad \beta^2 = \frac{a}{4\sqrt{1-m+m^2}}, \quad B = \frac{a[2-m-\sqrt{1-m+m^2}]}{2b\sqrt{1-m+m^2}}. \quad (124)$$

Remarkably, we find that the same model also admits the superposed periodic solution

$$\phi = \frac{A}{2} \operatorname{dn}^2[\beta(x+c), m] + \frac{D}{2} \sqrt{m} \operatorname{cn}[\beta(x+c), m] \operatorname{dn}[\beta(x+c), m] + B, \quad (125)$$

provided

$$D = A = -\frac{6a}{b\sqrt{1+14m+m^2}}, \quad \beta^2 = \frac{a}{\sqrt{1+14m+m^2}}, \quad B = \frac{a[5-m-\sqrt{1+14m+m^2}]}{2b\sqrt{1+14m+m^2}}. \quad (126)$$

6 Coupled Field Theories with $\operatorname{dn}^2 \pm \sqrt{m} \operatorname{cn} \operatorname{dn}$ Solutions

We now discuss three examples of coupled field theories which are known to admit dn^2 as a periodic solution in both the fields, and show that these coupled models also admit $\operatorname{dn}^2 \pm \sqrt{m} \operatorname{cn} \operatorname{dn}$ as solutions in both the fields.

6.1 Quadratic NLS-KdV Coupled Theory

We first consider the quadratic NLS-KdV (QNLS-KdV) coupled system. The field equations for the coupled QNLS-KdV system are

$$\begin{aligned} iu_t + u_{xx} + g_1|u|u + \alpha uv &= 0, \\ v_t + v_{xxx} + 6vv_x + \gamma v|u|_x &= 0, \end{aligned} \quad (127)$$

where u and v denote the QNLS and KdV fields, respectively. It is easily shown that this coupled system admits an exact solution

$$\begin{aligned} u &= \left(A \operatorname{dn}^2[\beta(x - ct + \delta_1), m] + D \right) \exp[-i(\omega t - kx + \delta)], \\ v &= B \operatorname{dn}^2[\beta(x - ct + \delta_1), m] + F, \end{aligned} \quad (128)$$

provided

$$gA + \alpha B = 6\beta^2, \quad \gamma A + 6B = 12\beta^2, \quad (129)$$

and further

$$c = 2k, \quad \omega = k^2 - 2[2(2 - m) + 3(z + y)] + Ag(z - y), \quad c = 4[2 - m + 3z]\beta^2, \quad (130)$$

with

$$z = \frac{F}{B}, \quad y = \frac{D}{A} = \frac{-(2 - m) \pm \sqrt{m^2 - 2(1 - m)}}{3}. \quad (131)$$

On solving the relations (129), we find that in general A, B are given by

$$A = \frac{12(3 - \alpha)\beta^2}{6g - \alpha\gamma}, \quad B = \frac{6(2g - \gamma)\beta^2}{6g - \alpha\gamma}. \quad (132)$$

Only in the special case when $\gamma = 2g, \alpha = 3$ that A, B cannot be separately determined but they only satisfy the constraint

$$3B + gA = 6\beta^2. \quad (133)$$

Remarkably, even a superposition, i.e.

$$\begin{aligned} u &= \left(D + \frac{A}{2} \operatorname{dn}^2[\beta(x - ct + \delta_1), m] \right. \\ &\quad \left. + \frac{G}{2} \sqrt{m} \operatorname{cn}[\beta(x - ct + \delta_1), m] \operatorname{dn}[\beta(x - vt + \delta_1), m] \right) \exp[-i(\omega t - kx + \delta)], \end{aligned} \quad (134)$$

and

$$\begin{aligned} v &= F + \frac{B}{2} \operatorname{dn}^2[\beta(x - ct + \delta_1), m] \\ &\quad + \frac{H}{2} \sqrt{m} \operatorname{cn}[\beta(x - ct + \delta_1), m] \operatorname{dn}[\beta(x - ct + \delta_1), m], \end{aligned} \quad (135)$$

is an exact solution to Eqs. (127) provided relations (129) (and hence (132) or (133) are satisfied) and further

$$\begin{aligned} c = 2k, G = \pm A, \quad H = \pm B, \quad c = [5 - m + 12z], \quad z = \frac{F}{B}, \\ \omega = k^2 - [5 - m + 6(z + y)]\beta^2 + gA(z - y), \quad y = \frac{D}{A} = \frac{-(5 - m) \pm \sqrt{1 + 14m + m^2}}{12}. \end{aligned} \quad (136)$$

Note that the signs of $H = \pm B$ and $G = \pm A$ are correlated.

6.2 NLS-MKdV Coupled Field Theory

In section III we have discussed coupled NLS-MKdV system and shown that it admits cn, dn as well as superposed solutions of the form $\text{dn} \pm \sqrt{m}\text{cn}$ in both the fields. We now show that remarkably, the same coupled model not only admits dn^2 as a solution in both the fields but it also admits $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ as a solution in both the fields even though neither dn^2 nor cndn nor $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ is an exact solution of either of the uncoupled NLS or MKdV models.

The field equations for the coupled NLS-MKdV system as given in Sec. III (see Eqs. (53)) are

$$\begin{aligned} iu_t + u_{xx} + g_1|u|^2u + \alpha uv^2 &= 0, \\ v_t + v_{xxx} + 6v^2v_x + \gamma v(|u|^2)_x &= 0, \end{aligned} \quad (137)$$

where u and v denote the NLS and the MKdV fields, respectively. It is easily shown that this coupled system admits an exact solution

$$\begin{aligned} u &= \left(A \text{dn}^2[\beta(x - ct + \delta_1), m] + F \right) \exp[-i(\omega t - kx + \delta)], \\ v &= B \text{dn}^2[\beta(x - ct + \delta_1), m] + D, \end{aligned} \quad (138)$$

provided

$$\begin{aligned} \alpha = 3/2, \quad \gamma A^2 = -3B^2, \quad \gamma = 2g < 0, \quad (z - y)B^2 = 2\beta^2, \\ c = 2k = 4[(2 - m) + 3z]\beta^2, \quad \omega = k^2 - [4(2 - m) + 9y + 3z]\beta^2, \\ z = \frac{D}{B}, \quad y = \frac{F}{A} = \frac{[-(2 - m) \pm \sqrt{1 - m + m^2}]}{3}. \end{aligned} \quad (139)$$

Note that $A\text{dn}^2 + B$ is not a solution of either the NLS or the MKdV uncoupled field equations even though the coupled system admits such a solution in both the fields.

Remarkably, even a superposition, i.e.

$$u = \left(F + \frac{A}{2}\text{dn}^2[\beta(x - ct + \delta_1), m] + \frac{G}{2}\sqrt{m}\text{cn}[\beta(x - ct + \delta_1), m]\text{dn}[\beta(x - vt + \delta_1), m] \right) \exp[-i(\omega t - kx + \delta)], \quad (140)$$

and

$$v = D + \frac{B}{2}\text{dn}^2[\beta(x - ct + \delta_1), m] + \frac{H}{2}\sqrt{m}\text{cn}[\beta(x - ct + \delta_1), m]\text{dn}[\beta(x - ct + \delta_1), m], \quad (141)$$

is an exact solution to Eq. (137) provided

$$G = \pm A, \quad H = \pm B, \quad c = 2k = [5 - m + 12z]\beta^2, \\ \omega = k^2 - [(5 - m) + 9y + 3z]\beta^2, \quad y = \frac{F}{A} = \frac{[-(5 - m) \pm \sqrt{1 + 14m + m^2}]}{12}, \quad (142)$$

while rest of the relations are exactly the same as those given by Eq. (139). Again the signs of $G = \pm A$ and $H = \pm B$ are correlated.

6.3 Coupled NLS Models

In section III we have discussed a coupled NLS system and shown that it admits cn, dn as well as superposed solutions of the form $\text{dn} \pm \sqrt{m}\text{cn}$ in both the fields. We now show that remarkably, the same coupled model also admits dn^2 type as well as $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ type solutions in both the fields even though neither dn^2 nor cndn nor $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ is an exact solution of the uncoupled NLS equation.

The field equations for the coupled NLS system as given in Sec. III (see Eqs. (62)) are

$$iu_t + u_{xx} + [a|u|^2 + b|v|^2]u = 0, \\ v_t + v_{xx} + [f|u|^2 + e|v|^2]v = 0, \quad (143)$$

where u and v are the two coupled NLS fields.

It is easily checked that

$$u(x, t) = \exp[-i(\omega_1 t - k_1 x + \delta_1)](A \operatorname{dn}^2[\beta(x - ct + \delta), m] + D), \quad (144)$$

and

$$v(x, t) = \exp[-i(\omega_2 t - k_2 x + \delta_2)](B \operatorname{dn}^2[\beta(x - ct + \delta), m] + E), \quad (145)$$

is an exact solution to the coupled field equations (143) provided

$$c = 2k_1 = 2k_2, \quad a = f, \quad b = e, \quad aA^2 = -bB^2, \quad (z_{\pm} - y_{\mp})aA^2 = 6\beta^2, \quad z = \frac{D}{A}, \quad y = \frac{E}{B}, \quad (146)$$

$$\begin{aligned} z_{\pm} = y_{\pm} &= \frac{-(2-m) \pm \sqrt{1-m+m^2}}{3}, \\ \omega_1 - k_1^2 &= \mp 2\sqrt{1-m+m^2}\beta^2, \quad \omega_2 - k_1^2 = \pm 2\sqrt{1-m+m^2}\beta^2. \end{aligned} \quad (147)$$

From the relation (146) it follows that this solution exists only if z and y are unequal. Also, depending on whether we choose z_+, y_- or z_-, y_+ , the corresponding ω_1, ω_2 are as given by Eq. (147).

Note that this solution is only valid if a, b have opposite signs, hence this solution can only be valid in the MZS case [21, 22, 23] but not in the Manakov case [15].

Remarkably, it turns out that even a superposition of dn^2 and $\operatorname{cn} \operatorname{dn}$ is an exact solution to the coupled Eqs. (143). In particular, it is easily checked that

$$\begin{aligned} u(x, t) &= \exp[-i(\omega_1 t - kx + \delta_1)] \left(\frac{A}{2} \operatorname{dn}^2[\beta(x - ct + \delta), m] \right. \\ &\quad \left. + D + \frac{G}{2} \sqrt{m} \operatorname{cn}[\beta(x - ct + \delta), m] \operatorname{dn}[\beta(x - ct + \delta), m] \right), \end{aligned} \quad (148)$$

and

$$\begin{aligned} v(x, t) &= \exp[-i(\omega_2 t - kx + \delta_1)] \left(\frac{B}{2} \operatorname{dn}^2[\beta(x - ct + \delta), m] \right. \\ &\quad \left. + E + \frac{H}{2} \sqrt{m} \operatorname{cn}[\beta(x - ct + \delta), m] \operatorname{dn}[\beta(x - ct + \delta), m] \right), \end{aligned} \quad (149)$$

is an exact solution to the coupled field equations (143) provided

$$\begin{aligned} c = 2k_1 = 2k_2, \quad G = \pm A, \quad H = \pm B, \quad a = f, \quad b = e, \quad aA^2 = -bB^2, \\ (z_{\pm} - y_{\mp})aA^2 = 6\beta^2, \quad z = \frac{D}{A}, \quad y = \frac{E}{B}, \end{aligned} \quad (150)$$

$$\begin{aligned}
z_{\pm} = y_{\pm} &= \frac{-(5-m) \pm \sqrt{1+14m+m^2}}{12}, \\
\omega_1 - k_1^2 &= \mp \frac{\sqrt{1-m+m^2}}{2} \beta^2, \quad \omega_2 - k_2^2 = \pm \frac{\sqrt{1-m+m^2}}{2} \beta^2.
\end{aligned} \tag{151}$$

Thus like the dn^2 solution, this solution exists only if z and y are unequal. Also, depending on whether we choose z_+, y_- or z_-, y_+ , the corresponding ω_1, ω_2 are as given by Eq. (151). Further, like the last solution, this solution is only valid if a, b have opposite signs, hence this solution can only be valid in the MZS case [21, 22, 23] but not in the Manakov case [15].

7 Mixed $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ and $\text{dn} \pm \sqrt{m}\text{cn}$ Solutions in Coupled Field Theories

We consider four coupled field theory models in which one field has a solution in terms of either cn or dn while the other field admits a solution in terms of dn^2 . We now show that these models also admit superposed solutions of the form $\text{dn} \pm \sqrt{m}\text{cn}$ in one field and $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ in the other field.

7.1 Coupled NLS-KdV Fields

Let us consider the following coupled NLS-KdV field equations

$$\begin{aligned}
iu_t + u_{xx} + g|u|^2u + \alpha uv &= 0, \\
v_t + v_{xxx} + 6vv_x + \gamma v(|u|^2)_x &= 0,
\end{aligned} \tag{152}$$

where u and v are the NLS and the KdV fields, respectively. It is worth pointing out that this coupled model has been popular in the literature in the context of interaction (between a short wave and a long wave) in fluid mechanics and plasma physics [30]. We first show that these coupled equations admit solutions in terms of dn and cn for the NLS field and dn^2 for the KdV field.

It is easily checked that

$$u(x, t) = A \exp[-i(\omega t - kx + \delta_1)] \text{dn}[\beta(x - ct + \delta), m], \tag{153}$$

and

$$v(x, t) = B \text{dn}^2[\beta(x - ct + \delta_1), m] + D, \tag{154}$$

is an exact solution to the coupled Eqs. (152) provided

$$gA^2 + \alpha B = 2\beta^2, \quad \gamma A^2 + 6B = 12\beta^2, \quad (155)$$

$$c = 2k = 4[2 - m + 3z]\beta^2, \quad \omega = k^2 - (2 - m)\beta^2 - \alpha z B, \quad z = \frac{D}{B}. \quad (156)$$

On solving Eqs. (155) we find that

$$A^2 = \frac{12(\alpha - 1)\beta^2}{\alpha\gamma - 6g}, \quad B = \frac{2(\gamma - 6g)\beta^2}{(\alpha\gamma - 6g)}. \quad (157)$$

In the special case when $\alpha = 1, \beta = 6g$, A and B are undetermined and instead of the relations (157), we only have the constraint

$$B + gA^2 = 2\gamma^2. \quad (158)$$

It is easily checked that

$$u(x, t) = A \exp[-i(\omega t - kx + \delta)] \sqrt{m} \text{cn}[\gamma(x - ct + \delta_1), m], \quad (159)$$

and

$$v(x, t) = B \text{dn}^2[\gamma(x - ct + \delta_1), m] + D, \quad (160)$$

is an exact solution to the coupled field equations (152) provided

$$\omega = k^2 - (2m - 1)\gamma^2 - [\alpha z + (1 - m)]B, \quad (161)$$

while all other relations are exactly as given by Eqs. (155) to (158).

Remarkably, the same model also admits interesting superposed solutions of the form $\text{dn} \pm \sqrt{m} \text{cn}$ in the NLS field and $\text{dn}^2 \pm \sqrt{m} \text{cn} \text{dn}$ in the KdV field.

It is easily checked that

$$u(x, t) = \frac{1}{2} \exp[-i(\omega t - kx + \delta)] \left(A \text{dn}[\gamma(x - ct + \delta_1), m] + H \sqrt{m} \text{cn}[\gamma(x - ct + \delta_1), m] \right), \quad (162)$$

and

$$v(x, t) = \frac{B}{2} \text{dn}^2[\gamma(x - ct + \delta_1), m] + \frac{F}{2} \sqrt{m} \text{cn}[\gamma(x - ct + \delta_1), m] \text{dn}[\gamma(x - ct + \delta_1), m] + D, \quad (163)$$

is an exact solution to the coupled field equations (152) provided

$$\begin{aligned} z &= \frac{D}{B}, \quad H = \pm A, \quad F = \pm B, \quad c = 2k = [(5 - m) + 12z]\gamma^2, \\ \omega &= k^2 - \frac{(1 + m)\gamma^2}{2} - \frac{\alpha}{4}[(1 - m) + 4z]B, \end{aligned} \quad (164)$$

and further A, B are again either given by Eqs. (157) or satisfy the constraint (158). Note that the signs of $F = \pm B$ and $H = \pm A$ are correlated.

7.2 KdV-MKdV coupled System

Let us consider the following coupled KdV-MKdV field equations

$$\begin{aligned} u_t + u_{xxx} + 6uu_x + 2\alpha uvv_x &= 0, \\ v_t + v_{xxx} + 6v^2v_x + \gamma vu_x &= 0, \end{aligned} \quad (165)$$

where u and v are the KdV and the MKdV fields, respectively. We first show that these coupled equations admit dn^2 - dn and dn^2 - cn solutions.

It is easily checked that

$$u(x, t) = A \text{dn}^2[\beta(x - ct + \delta_1), m] + B, \quad (166)$$

and

$$v(x, t) = D \text{dn}[\beta(x - ct + \delta_1), m], \quad (167)$$

is an exact solution to the coupled field equations (165) provided

$$6A + \alpha D^2 = 12\beta^2, \quad \gamma A + 3D^2 = 3\beta^2, \quad (168)$$

and further

$$c = (2 - m)\beta^2, \quad \frac{B}{A} = -\frac{2 - m}{4}. \quad (169)$$

On solving Eqs. (168) we obtain

$$D^2 = \frac{6(2\gamma - 3)\beta^2}{\alpha\gamma - 18}, \quad A = \frac{3(\alpha - 12)\beta^2}{\alpha\gamma - 18}. \quad (170)$$

In the special case when $\gamma = 3/2, \alpha = 12$, A and D are undetermined and instead of the relations (170), we only have the constraint

$$2D^2 + A = 2\beta^2. \quad (171)$$

It is easily checked that

$$u(x, t) = A \operatorname{dn}^2[\beta(x - ct + \delta_1), m] + B, \quad (172)$$

and

$$v(x, t) = D\sqrt{m} \operatorname{cn}[\beta(x - ct + \delta_1), m], \quad (173)$$

is an exact solution to the coupled field equations (165) provided A, D satisfy Eq. (170) or the constraint (171), while the other two relations now are

$$c = (2m - 1)\beta^2, \quad \frac{B}{A} = -\frac{3 - 2m}{4}. \quad (174)$$

We now show that remarkably, these coupled equations admit even a linear superposition of the two, i.e. $\operatorname{dn}^2 \pm \sqrt{m} \operatorname{cn} \operatorname{dn}$ as a solution of the KdV field and $\operatorname{dn} \pm \sqrt{m} \operatorname{cn}$ as solution of the MKdV field. It is easily checked that

$$\begin{aligned} u(x, t) &= \frac{A}{2} \operatorname{dn}^2[\beta(x - ct + \delta_1), m] \\ &+ \frac{F}{2} \sqrt{m} \operatorname{cn}[\beta(x - ct + \delta_1), m] \operatorname{dn}[\beta(x - ct + \delta_1), m] + B, \end{aligned} \quad (175)$$

and

$$v(x, t) = \frac{D}{2} \operatorname{dn}[\beta(x - ct + \delta_1), m] + \frac{G}{2} \sqrt{m} \operatorname{cn}[\beta(x - ct + \delta_1), m], \quad (176)$$

is an exact solution to the coupled field equations (165) provided

$$F = \pm A, \quad G = \pm D, \quad c = \frac{(1 + m)}{2} \beta^2, \quad \frac{B}{A} = -\frac{3 - m}{8}, \quad (177)$$

while A, D satisfy the relations (170) or the constraint (171). Note that the signs of $F = \pm A$ and $G = \pm D$ are correlated.

7.3 Quadratic NLS-MKdV Coupled Model

Let us consider a coupled QNLS-MKdV model with the field equations

$$\begin{aligned} iu_t + u_{xx} + g|u|u + \alpha uv^2 &= 0, \\ v_t + v_{xxx} + 6v^2v_x + \gamma v|u|_x &= 0, \end{aligned} \quad (178)$$

where u and v are the QNLS and the MKdV fields, respectively. We first show that these coupled equations admit dn^2 - dn and dn^2 - cn solutions.

It is easily checked that

$$u(x, t) = \left(A \text{dn}^2[\beta(x - ct + \delta_1), m] + B \right) e^{-i[\omega t - kx + \delta]}, \quad (179)$$

and

$$v(x, t) = D \text{dn}[\beta(x - ct + \delta_1), m], \quad (180)$$

is an exact solution to the coupled field equations (178) provided

$$\alpha D^2 + gA = 6\beta^2, \quad 3D^2 + \gamma A = 3\beta^2, \quad (181)$$

and further

$$\begin{aligned} c = 2k = (2 - m)\beta^2, \quad z = \frac{B}{A} &= \frac{-(2 - m) \pm \sqrt{1 - m + m^2}}{3}, \\ \omega = k^2 - 2[2(2 - m) + 3z]\beta^2 - Agz. \end{aligned} \quad (182)$$

On solving Eqs. (181) we find that A, D^2 are given by

$$A = \frac{3(6 - \alpha)\beta^2}{3g - \alpha\gamma}, \quad D^2 = \frac{3(g - 2\gamma)\beta^2}{3g - \alpha\gamma}. \quad (183)$$

However, in the special case when $2\gamma = g, \alpha = 6$, A and D are undetermined and instead of the relations (183), we only have the constraint

$$3D^2 + \gamma A = 3\beta^2. \quad (184)$$

It is easily checked that

$$u(x, t) = \left(A \text{dn}^2[\beta(x - ct + \delta_1), m] + B \right) e^{-i[\omega t - kx + \delta]}, \quad (185)$$

and

$$v(x, t) = D\sqrt{m}\text{cn}[\beta(x - ct + \delta_1), m], \quad (186)$$

is an exact solution to the coupled field equations (178) provided A, D satisfy relations (183) or the constraint (184), while the other relations now are

$$\begin{aligned} c = 2k = (2m - 1)\beta^2, \quad z = \frac{B}{A} = \frac{-(2 - m) \pm \sqrt{1 - m + m^2}}{3}, \\ \omega = k^2 - 2[1 + m + 3z]\beta^2 - Ag[1 - m + z]. \end{aligned} \quad (187)$$

We now show that remarkably, the coupled model also admits even a linear superposition of the two as an exact solution, i.e. $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ as solution of the quadratic NLS field and $\text{dn} \pm \sqrt{m}\text{cn}$ as solution of the MKdV field. It is easily checked that

$$\begin{aligned} u(x, t) = & \left(\frac{A}{2} \text{dn}^2[\beta(x - ct + \delta_1), m] \right. \\ & \left. + \frac{F}{2} \sqrt{m}\text{cn}[\beta(x - ct + \delta_1), m] \text{dn}[\beta(x - ct + \delta_1), m] + B \right) e^{-i[\omega t - kx + \delta]}, \end{aligned} \quad (188)$$

and

$$v(x, t) = \frac{D}{2} \text{dn}[\beta(x - ct + \delta_1), m] + \frac{G}{2} \sqrt{m}\text{cn}[\beta(x - ct + \delta_1), m], \quad (189)$$

is an exact solution to the coupled field equations (178) provided

$$\begin{aligned} F = \pm A, \quad G = \pm D, \quad z = \frac{B}{A} = -\frac{5 - m \pm \sqrt{1 + 14m + m^2}}{12}, \\ c = \frac{(1 + m)}{2} \beta^2, \quad \omega = k^2 - \left[\frac{7 + m}{2} + 6z \right] - gA \left[z + \frac{1 - m}{4} \right], \end{aligned} \quad (190)$$

while A, D satisfy the relations (183) or the constraint (184). Note that the signs of $F = \pm A$ and $G = \pm D$ are correlated.

7.4 Quadratic NLS-NLS coupled Model

Let us consider the following coupled QNLS-NLS model with field equations

$$\begin{aligned} iu_t + u_{xx} + g_1|u|u + \alpha u|v|^2 &= 0, \\ iv_t + v_{xx} + g_2|v|^2v_x + \gamma v|u| &= 0, \end{aligned} \quad (191)$$

where u and v are the QNLS and the NLS fields, respectively. We first show that these coupled equations admit dn^2 as solution of the quadratic NLS and either dn or cn as the solution of the NLS field.

It is easily checked that

$$u(x, t) = \left(A \text{dn}^2[\beta(x - ct + \delta_1), m] + B \right) e^{-i[\omega_1 t - kx + \delta]}, \quad (192)$$

and

$$v(x, t) = D \text{dn}[\beta(x - ct + \delta_1), m] e^{-i[\omega_2 t - k_1 x + \delta_2]}, \quad (193)$$

is an exact solution to the coupled field Eqs. (191) provided

$$\alpha D^2 + g_1 A = 6\beta^2, \quad g_2 D^2 + \gamma A = 2\beta^2, \quad (194)$$

and further

$$\begin{aligned} c = 2k = 2k_1, \quad z = \frac{B}{A} &= \frac{-(2-m) \pm \sqrt{1-m+m^2}}{3}, \\ \omega_1 = k^2 - 2[2(2-m) + 3z]\beta^2 - Ag_1 z, \quad \omega_2 &= k^2 - (2-m)\beta^2 - A\gamma z. \end{aligned} \quad (195)$$

On solving Eqs. (194) we find that A, D^2 are given by

$$A = \frac{2(\alpha - 3g_2)\beta^2}{\alpha\gamma - g_1 g_2}, \quad D^2 = \frac{2(3\gamma - g_1)\beta^2}{\alpha\gamma - g_1 g_2}. \quad (196)$$

However, in the special case when $3\gamma = g_1, \alpha = 3g_2$, A and D are undetermined and instead of the relations (196), we only have the constraint

$$3g_2 D^2 + g_1 A = 6\beta^2. \quad (197)$$

It is easily checked that

$$u(x, t) = \left(A \text{dn}^2[\beta(x - ct + \delta_1), m] + B \right) e^{-i[\omega_1 t - kx + \delta]}, \quad (198)$$

and

$$v(x, t) = D \sqrt{m} \text{cn}[\beta(x - ct + \delta_1), m] e^{-i[\omega_2 t - k_1 x + \delta_2]}, \quad (199)$$

is an exact solution to the coupled field equations (191) provided the relations (196) are or the constraint (197) is satisfied and further

$$\begin{aligned} c = 2k = 2k_1, \quad z = \frac{B}{A} &= \frac{-(2-m) \pm \sqrt{1-m+m^2}}{3}, \\ \omega_1 = k^2 - 2(1+m+3z)\beta^2 - Ag_1(z+1-m), \quad \omega_2 &= k^2 - (2m-1)\beta^2 - A\gamma(z+1-m). \end{aligned} \quad (200)$$

We now show that remarkably, the model also admits even a linear superposition of the two as exact solutions, i.e. $\text{dn}^2 \pm \sqrt{m}\text{cn}\text{dn}$ as a solution of the quadratic NLS field and $\text{dn} \pm \sqrt{m}\text{cn}$ as the solution of the NLS field. It is easily checked that

$$u(x, t) = \left(\frac{A}{2} \text{dn}^2[\beta(x - ct + \delta_1), m] + \frac{F}{2} \sqrt{m} \text{cn}[\beta(x - ct + \delta_1), m] \text{dn}[\beta(x - ct + \delta_1), m] + B \right) e^{-i[\omega_1 t - kx + \delta]}, \quad (201)$$

and

$$v(x, t) = \left(\frac{D}{2} \text{dn}[\beta(x - ct + \delta_1), m] + \frac{G}{2} \sqrt{m} \text{cn}[\beta(x - ct + \delta_1), m] \right) e^{-i[\omega_2 t - k_1 x + \delta_2]}, \quad (202)$$

is an exact solution to the coupled field equations (191) provided

$$\begin{aligned} F = \pm A, \quad G = \pm D, \quad z = \frac{B}{A} = -\frac{5 - m \pm \sqrt{1 + 14m + m^2}}{12}, \\ c = 2k = 2k_1, \quad \omega_1 = k^2 - \left[\frac{7 + m}{2} + 6z \right] - gA \left[z + \frac{1 - m}{4} \right], \\ \omega_2 = k^2 - \frac{1 + m}{2} \beta^2 - \gamma A \left(z + \frac{1 - m}{4} \right), \end{aligned} \quad (203)$$

while A, D satisfy the relations (196) or the constraint (197). Again note that the signs of $F = \pm A$ and $G = \pm D$ are correlated.

8 Summary and Conclusions

In this paper we have demonstrated through many examples that a kind of linear superposition holds good in the case of several discrete as well as continuum nonlinear equations. In particular, as a support to our conjecture, we have presented several nonlinear equations which admit $\text{dn}(x, m)$ and $\text{cn}(x, m)$ as periodic solutions, and have shown that all of them, without fail, also admit $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$ as periodic solutions. We would like to reemphasize that what we have proposed is only a kind of linear superposition and not the full linear superposition that is obtained in the linear theories. However, we find it remarkable that in spite of the nonlinear terms, even a kind of linear superposition seems to hold good in many nonlinear models, both discrete and continuum. We have also shown that such superposed solutions also exist in several *coupled* field theories. Many of these models such as MKdV, NLS, saturable discrete

NLS, etc. have found wide ranging applications in many interesting physics problems [5, 6, 12, 25, 30]. While some of these models are integrable, many others are nonintegrable. It would be worth enquiring whether the new solutions we have obtained have specific physical relevance. Thus, it would be important to examine the (linear and nonlinear) stability of these newly found solutions. In this context it is worth pointing out that the stability of the cn and dn solutions in the NLS and MKdV equations has been examined in detail in recent years. Inspired by the earlier work of Rowlands [31], it has been shown that in the NLS case, the cn solution is unstable [32]. However, in the MKdV case, it has been shown that the dn solution is stable with respect to the periodic perturbations of the same period and that this is true for all values of the modulus parameter m . On the other hand, the stability of the cn solution changes as the value of m changes [33, 34]. Stability analysis has also been done for the cn and dn solutions of the saturable discrete nonlinear Schrödinger equation [12] and it has been shown that both of these solutions are stable over a sizable amount of the parameter space.

Similarly we have also shown that several nonlinear equations which admit $\text{dn}^2(x, m)$ as a solution, also admit $\text{dn}^2(x, m) \pm \sqrt{m}\text{cn}(x, m)\text{dn}(x, m)$ type superposed solution. While this cannot be regarded as a linear superposition, since $\text{cn}(x, m)\text{dn}(x, m)$ is not a solution of these models, we find it rather remarkable that such solutions, without fail, hold good in several nonlinear equations. Further, we have also shown the existence of such solutions in many coupled systems which admit $\text{dn}^2(x, m)$ as a solution in both the fields. Since such solutions occur in physically important models like KdV, quadratic NLS, etc. [5, 6] it would be insightful to check the (linear as well as nonlinear) stability of such solutions. In this context it is worth pointing out that the stability of the dn^2 solution has received some attention in the literature. Inspired by the early works of Benjamin [35] and Bona [36], It has been shown [37] that the dn^2 solution is stable with respect to perturbation of an arbitrary period or even with respect to perturbations that are quasi-periodic. As far as dn^2 solution of the ϕ^2 - ϕ^3 field theory is concerned, one would expect it to be unstable. Nevertheless, it would be worthwhile to explicitly check the stability of the dn^2 as well as $\text{dn}^2 \pm \sqrt{m}\text{cndn}$ solutions in this case.

In addition, we have also considered several coupled theories which admit $\text{cn}(x, m)$ and $\text{dn}(x, m)$ type solutions in one field and $\text{dn}^2(x, m)$ type solutions in the other field and shown that these models also

admit superposed solutions of the form $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)\text{dn}(x, m)$ in the first field and $\text{dn}^2(x, m) \pm \sqrt{m}\text{cn}(x, m)\text{dn}(x, m)$ in the second field. Again, it is of physical importance to check the stability of these solutions.

What could be the possible reason why a linear superposition of the form $\text{dn}(x, m) \pm \sqrt{m}\text{cn}(x, m)$ is a solution for several nonlinear equations which admit $\text{cn}(x, m)$ and $\text{dn}(x, m)$ as solutions? We believe that the main reason is that $\text{cn}(x, m)$ and $\text{dn}(x, m)$ functions are quite similar, both being an even function of its argument and both of them as well as their derivatives being identical at $m = 1$. This is in contrast to the $\text{sn}(x, m)$ Jacobi elliptic function which is an odd function of its argument and at $m = 1$ it goes to $\tanh(x)$. This is in contrast to the even functions $\text{cn}(x, m)$ and $\text{dn}(x, m)$, both of which at $m = 1$, go over to $\text{sech}(x)$, and that is why $\text{cn}(x, m) \pm \text{sn}(x, m)$ superposition does not seem to work in the several examples that we have checked.

Presumably there is a deeper reason to all this which needs to be explored. It would be worthwhile to find a rigorous proof of our conjecture. It would also be interesting if one can find a counter-example to our conjecture. One line of thought is that the solutions are all of the traveling wave form and many of the PDEs can be reduced to an ODE either of the form $u + u_{xx} + u^2 = c$ or $u + u_{xx} + u^3 = 0$. Understanding these two simple ODEs might shed some light on the underlying general principle for the kind of superposition we have found. Note that in both cases the superposed solutions exist; however, at present we are unable to draw any general conclusions about the proposed superposition.

9 Acknowledgement

This work was supported in part by the U.S. Department of Energy.

10 Appendix: Dew Local Identities for Jacobi Elliptic Functions

In this Appendix, we list the various local identities for Jacobi elliptic functions [29] which have been used while deriving the various discrete solutions in Sec. IV.

$$\operatorname{dn}^2(x, m)[\operatorname{dn}(x+a, m) + \operatorname{dn}(x-a, m)] = 2\operatorname{ns}(a, m)\operatorname{ds}(a, m)\operatorname{dn}(x, m) - \operatorname{cs}^2(a, m)[\operatorname{dn}(x+a, m) + \operatorname{dn}(x-a, m)], \quad (204)$$

$$m\operatorname{cn}^2(x, m)[\operatorname{cn}(x+a, m) + \operatorname{cn}(x-a, m)] = 2\operatorname{ns}(a, m)\operatorname{cs}(a, m)\operatorname{cn}(x, m) - \operatorname{ds}^2(a, m)[\operatorname{cn}(x+a, m) + \operatorname{cn}(x-a, m)], \quad (205)$$

$$\operatorname{dn}^2(x, m)[\operatorname{dn}(x+a, m) - \operatorname{dn}(x-a, m)] = -2m\operatorname{cs}(a, m)\operatorname{cn}(x, m)\operatorname{sn}(x, m) - \operatorname{cs}^2(a, m)[\operatorname{dn}(x+a, m) - \operatorname{dn}(x-a, m)], \quad (206)$$

$$m\operatorname{cn}^2(x, m)[\operatorname{cn}(x+a, m) - \operatorname{cn}(x-a, m)] = -2\operatorname{ds}(a, m)\operatorname{sn}(x, m)\operatorname{dn}(x, m) - \operatorname{ds}^2(a, m)[\operatorname{cn}(x+a, m) - \operatorname{cn}(x-a, m)], \quad (207)$$

$$\begin{aligned} \operatorname{cn}(x, m)\operatorname{dn}(x, m)[\operatorname{dn}(x+a, m) + \operatorname{dn}(x-a, m)] = \\ 2\operatorname{ds}(a, m)\operatorname{ns}(a, m)\operatorname{cn}(x, m) - \operatorname{cs}(a, m)\operatorname{ds}(a, m)[\operatorname{cn}(x+a, m) + \operatorname{cn}(x-a, m)], \end{aligned} \quad (208)$$

$$\begin{aligned} \operatorname{cn}(x, m)\operatorname{dn}(x, m)[\operatorname{dn}(x+a, m) - \operatorname{dn}(x-a, m)] = \\ -2\operatorname{cs}(a, m)\operatorname{sn}(x, m)\operatorname{dn}(x, m) - \operatorname{cs}(a, m)\operatorname{ds}(a, m)[\operatorname{cn}(x+a, m) - \operatorname{cn}(x-a, m)], \end{aligned} \quad (209)$$

$$\begin{aligned} m\operatorname{cn}(x, m)\operatorname{dn}(x, m)[\operatorname{cn}(x+a, m) + \operatorname{cn}(x-a, m)] = 2\operatorname{cs}(a, m)\operatorname{ns}(a, m)\operatorname{dn}(x, m) \\ - \operatorname{cs}(a, m)\operatorname{ds}(a, m)[\operatorname{dn}(x+a, m) + \operatorname{dn}(x-a, m)], \end{aligned} \quad (210)$$

$$\begin{aligned} m\operatorname{cn}(x, m)\operatorname{dn}(x, m)[\operatorname{cn}(x+a, m) - \operatorname{cn}(x-a, m)] = -2\operatorname{ds}(a, m)\operatorname{sn}(x, m)\operatorname{cn}(x, m) \\ - \operatorname{cs}(a, m)\operatorname{ds}(a, m)[\operatorname{dn}(x+a, m) - \operatorname{dn}(x-a, m)], \end{aligned} \quad (211)$$

References

- [1] P.G. Kevrekidis, K.Ø. Rasmussen, and A.R. Bishop, Int. J. Mod. Phys. B **15**, 2833 (2001).

- [2] S. Flach and A.V. Gorbach, Phys. Rep. **467**, 1 (2008).
- [3] P.G. Kevrekidis, (Ed.), The Discrete Nonlinear Schrödinger Equation. Mathematical Analysis, Numerical Computations and Physical Perspectives, (Springer, Berlin, 2009).
- [4] See for example, M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions (Dover Publications, New York, 2010).
- [5] See for example, *Solitons: An Introduction*, by P.G. Drazin and R.S. Johnson and references therein (Cambridge Univ. Press, 1989)
- [6] *Physics of Solitons*, by T. Dauxois and M. Peyrard (Cambridge Univ. Press, 2006).
- [7] *Solitons and Instantons*, by R. Rajaraman (North Holland, Amsterdam, 1982).
- [8] K. Hayata and M. Koshiba, Phys. Rev. E **50**, 3267 (1994).
- [9] J. Fujioka, E. Cortes, R. Perez-Pascual, R. F. Rodriguez, A. Espinosa, and B. A. Malomed, Chaos **21**, 033120 (2011).
- [10] M.J. Ablowitz and J.F. Ladik, J. Math. Phys. **17**, 1011 (1976).
- [11] S. Takeno, J. Phys. Soc. Jpn. **61**, 1433 (1992); Y. Xiao, Phys. Lett. A **193**, 419 (1994); X. F. Yang and R. Schmid, Phys. Lett. A **195**, 63 (1994).
- [12] A. Khare, K.O. Rasmussen, M.R. Samuelsen and A. Saxena, J. Phys. A **38**, 807 (2005).
- [13] S. Dmitriev, P.G. Kevrekidis, A. Khare, and A. Saxena, J. Phys. A **40**, 6267 (2007).
- [14] A. Khare and A. Saxena, J. Math. Phys. **47**, 092902 (2006).
- [15] S.V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 1392 (1973) [Sov. Phys. JETP **38**, 693 (1974)].
- [16] A.-K. Farid, Y. Yu, A. Saxena, and P. Kumar, Phys. Rev. B **71**, 104509 (2005).
- [17] B. Zwiebach, J. High Energy Phys. **09**, 028 (2000); J. A. Minahan and B. Zwiebach, J. High Energy Phys. **09**, 029 (2000).

- [18] A. Khare and A. Saxena, Phys. Lett. A **377**, 2761 (2013).
- [19] D. Bazeia, Braz. J. Phys. **32**, 869 (2002).
- [20] M. Sanati and A. Saxena, Am. J. Phys. **71**, 1005 (2003).
- [21] A.V. Mikhailov, Physica D **3**, 73 (1981).
- [22] V.E. Zakharov and E.I. Schulman, Physica D **4**, 270 (1982).
- [23] V.S. Gerdjikov, in *Proceedings of the sixth International Conference on Geometry, Integrability and Quantization*, Varna, Bulgaria, 3-10 June 2004, edited by I.M. Mladenov and A.C. Hirshfeld (softex, Sofia, 2005), pp. 1-48.
- [24] D. Cai, A.R. Bishop, and A. Sánchez, Phys. Rev. E **48**, 1447(1993).
- [25] S. Gatz and J. Hermann, J. Opt. Soc. Am. B **8**, 2296 (1991); Opt. Lett. **17**, 484 (1992).
- [26] A. Malukov, L. Hadzievski, B. A. Malomed, and L. Salasnich, Phys. Rev. A **78**, 013616 (2008).
- [27] F. Cooper, A. Khare, B. Mihaila, and A. Saxena, Phys. Rev. E **72** (2005) 036605.
- [28] A. Khare, A. Saxena and K. J. H. Law, J. Phys. A **42**, 475404 (2009).
- [29] A. Khare and U.P. Sukhatme, J. Math. Phys. **43**, 3798 (2002); A. Khare, A. Lakshminarayan, and U.P. Sukhatme, J. Math. Phys. **44**, 1822 (2003); A. Khare, A. Lakshminarayan, and U.P. Sukhatme, Pramana (J. Of Phys.) **62**, 1201 (2004).
- [30] See for example, E.S. Benilov and S.P. Burtsev, Phys. Lett. A **98**, 256 (1983); For comprehensive references, see for example, A. Arbieto, A. J. Corcho, and C. Matheus, J. Diff. Eqs. **230**, 295 (2006).
- [31] G. Rowlands, J. Inst. Maths. Applications. **13**, 367 (1974).
- [32] T. Ivey and S. Lafortune, Physica D **237**, 1750 (2008).
- [33] J. Angulo Pava, J. Diff. Eqs. **235**, 1 (2007).

- [34] M.A. Nivala, Ph.D. Thesis, University of Washington (2009).
- [35] T.B. Benjamin, Lectures on Nonlinear Wave Motion, ed. A. C. Newell, Amer. Math. Soc. (Providence, RI) **15**, 3 (1974).
- [36] J. Bona, Proc. Roy. Soc. London A **344**, 363 (1975).
- [37] N. Bottman and B. Deconinch, Discrete and Continuous Dynamical Systems A **25**, 1163 (2009).